Exercise 7

Use power series to solve the differential equation.

$$(x-1)y'' + y' = 0$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$(x-1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Expand the left side.

$$x\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Bring x inside the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Because of the $n-1$ inside the summand, the first series can start from $n = 1$.

$$\sum_{n=1}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Make the substitution $n = k$ in the first series, the substitution $n = k+1$ in the second series, and the substitution $n = k$ in the third series.

$$\sum_{k=1}^{\infty} k(k-1) a_k x^{k-1} - \sum_{k+1=2}^{\infty} (k+1)[(k+1)-1] a_{k+1} x^{(k+1)-2} + \sum_{k=1}^{\infty} k a_k x^{k-1} = 0$$

Simplify the second series.

$$\sum_{k=1}^{\infty} k(k-1) a_k x^{k-1} - \sum_{k=1}^{\infty} (k+1)k a_{k+1} x^{k-1} + \sum_{k=1}^{\infty} k a_k x^{k-1} = 0$$

Since all series start from $k = 1$ and have x^{k-1} in the summands, they can be combined.

$$\sum_{k=1}^{\infty} \left[k(k-1)a_k - (k+1)ka_{k+1} + ka_k\right] x^{k-1} = 0$$

The quantity in square brackets must be zero.

$$k(k-1)a_k - (k+1)ka_{k+1} + ka_k = 0$$

$$k^2 a_k - k(k+1)a_{k+1} = 0$$

Solve for a_{k+1}, noting that $1 \leq k < \infty$.

$$a_{k+1} = \frac{k}{k+1} a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$\begin{aligned}
k = 1: &\quad a_2 = \frac{1}{1+1} a_1 = \frac{1}{2} a_1 \\
k = 2: &\quad a_3 = \frac{2}{2+1} a_2 = \frac{2}{3}\left(\frac{1}{2} a_1\right) = \frac{1}{3} a_1 \\
k = 3: &\quad a_4 = \frac{3}{3+1} a_3 = \frac{3}{4}\left(\frac{1}{3} a_1\right) = \frac{1}{4} a_1 \\
&\quad \vdots
\end{aligned}$$

The general formula is

$$a_m = \frac{1}{m} a_1$$

for $1 \leq m < \infty$. Therefore, the general solution is

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= a_0 + \sum_{m=1}^{\infty} a_m x^m \\
&= a_0 + \sum_{m=1}^{\infty} \frac{1}{m} a_1 x^m \\
&= a_0 + a_1 \sum_{m=1}^{\infty} \frac{x^m}{m},
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. Note that for the infinite series to converge, $-1 < x < 1$.